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A note on the orthogonality equation with two functions

RADOSŁAW ŁUKASIK

Abstract. The aim of this paper is to describe the solution (f, g) of the equation

$$\langle f(x)|g(y) \rangle = \langle x|y \rangle, \quad x, y \in D,$$

where $f, g: D \rightarrow Y$, X, Y are Hilbert spaces over the same field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, D is a dense subspace of X .

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1. Introduction

Throughout this paper X, Y are Hilbert spaces over the same field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, $\langle \cdot | \cdot \rangle$ denotes the inner product and $\|\cdot\|$ the norm associated with it. We shall not distinguish between the symbols used for X and Y , $D(T)$ denotes the domain of the operator T .

It is known that $h: X \rightarrow Y$ is a solution of the orthogonality equation:

$$\langle h(x)|h(y) \rangle = \langle x|y \rangle, \quad x, y \in X$$

if and only if h is a linear isometry.

Chmieliński [2] studied the generalized orthogonality equation

$$\langle f(x)|g(y) \rangle = \langle x|y \rangle, \quad x, y \in X,$$

with two unknown functions $f, g: X \rightarrow Y$. The form of solutions of the above equation was presented by Łukasik and Wójcik [4]. We would like to present solutions in the case of inner product spaces.

We use some facts from the theory of adjoint operators.

Definition 1. Let $D(f)$ be a linear subspace of X , $f: D(f) \rightarrow Y$ be a linear operator. Further, let

$$D(f^*) = \left\{ y \in Y : \bigvee_{z \in X} \bigwedge_{x \in D(f)} \langle f(x)|y \rangle = \langle x|z \rangle \right\}.$$

A function $f^*: Y \rightarrow X$ is called an adjoint operator iff

$$\langle f(x)|y \rangle = \langle x|f^*(y) \rangle, \quad x \in D(f), \quad y \in D(f^*).$$

Remark 2 (See [1, § 39]). Let $D(f)$ be a dense linear subspace of X , $f: D(f) \rightarrow Y$ be a linear operator. Then the adjoint operator exists and it is unique. Moreover $D(f^*)$ is a linear subspace of Y , f^* is a linear operator and

$$\ker f^* = (\operatorname{im} f)^\perp, \quad \operatorname{cl} \operatorname{im} f = (\ker f^*)^\perp.$$

Definition 3. Let $D(f)$ be a linear subspace of X . A linear operator $f: D(f) \rightarrow Y$ is called bounded iff there exist $M > 0$ such that

$$\|f(x)\| \leq M \|x\|, \quad x \in D(f).$$

Remark 4 (See [1, § 19]). Let $D(f)$ be a dense linear subspace of X , $f: D(f) \rightarrow Y$ be a linear and bounded operator. Then f can be uniquely extended to a linear and bounded operator on X .

Definition 5. Let $D(f)$ be a dense linear subspace of X , $f: D(f) \rightarrow Y$ be a linear and injective operator. A function $f^{-1}: \operatorname{im} f \rightarrow X$ given by the formula

$$f^{-1}(y) = x, \quad y \in \operatorname{im} f, \quad \text{where } x \in D(f) \text{ satisfies } y = f(x)$$

is called an inversion of f .

Remark 6. The inversion of the linear and injective operator is linear and injective.

We also use some lemmas.

Lemma 7. (See [3]) Let $D \neq \emptyset$ be a set, $f, g: D \rightarrow Y$ be arbitrary mappings. Then there exist a subspace Y_0 of $\operatorname{cl} \operatorname{Lin} \operatorname{im} g$ and mappings $f_1, g_1: D \rightarrow Y_0$, $f_2: D \rightarrow (\operatorname{im} g)^\perp$, $g_2: D \rightarrow Y_0^\perp \cap \operatorname{cl} \operatorname{Lin} \operatorname{im} g$ such that

$$\begin{aligned} \langle f_1(x)|g_1(y) \rangle &= \langle f(x)|g(y) \rangle, \quad x, y \in D, \\ f &= f_1 + f_2, \quad g = g_1 + g_2, \\ \operatorname{cl} \operatorname{Lin} \operatorname{im} f_1 &= \operatorname{cl} \operatorname{Lin} \operatorname{im} g_1 = Y_0. \end{aligned}$$

Lemma 8. (See [2]) Let D be a dense linear subspace of X , $f, g: D \rightarrow Y$ satisfy the equation

$$\langle f(x)|g(y) \rangle = \langle x|y \rangle, \quad x, y \in D. \quad (1)$$

Then f and g are injective. Moreover, if $\operatorname{cl} \operatorname{im} g = Y$ then f is linear.

From the above lemmas we almost directly obtain

Theorem 9. Let D be a dense linear subspace of X , $f, g: D \rightarrow Y$ satisfy Eq. (1). Then there exist a subspace Y_0 of $\text{cl Lin im } g$, linear, injective mappings $f_1, g_1: D \rightarrow Y_0$ and maps $f_2: D \rightarrow (\text{im } g)^\perp$, $g_2: D \rightarrow Y_0^\perp \cap \text{cl Lin im } g$ such that

$$\begin{aligned}\langle f_1(x)|g_1(y) \rangle &= \langle x|y \rangle, \quad x, y \in D, \\ f &= f_1 + f_2, \quad g = g_1 + g_2, \\ \text{cl im } f_1 &= \text{cl im } g_1 = Y_0.\end{aligned}$$

In the main result we also use some lemma.

Lemma 10. Let D be a dense linear subspace of X , $T: X \rightarrow Y$ be a linear and injective operator such that $\text{cl im } T = Y$. Then

$$D \subset \text{im } T^* \iff D \subset D((T^{-1})^*).$$

Proof. (\implies) Assume that $D \subset \text{im } T^*$. Let $y \in D$, then $y = T^*(v)$ for some $v \in D(T^*)$. We have

$$\langle T^{-1}(T(x))|y \rangle = \langle x|y \rangle = \langle x|T^*(v) \rangle = \langle T(x)|v \rangle, \quad x \in D.$$

Hence $y \in D((T^{-1})^*)$.

(\impliedby) Assume that $D \subset D((T^{-1})^*)$. Since $T, T^{-1}, T^{-1} \circ T = Id_D$ are densely defined, $(Id_D)^* = (T^{-1} \circ T)^*$ is an extension of $T^* \circ (T^{-1})^*$. Hence $D = T^*((T^{-1})^*(D)) \subset \text{im } T^*$. \square

2. Main result

Theorem 11. Let D be a dense linear subspace of X . Then $f, g: D \rightarrow Y$ satisfy Eq. (1) iff there exist subspaces Y_0, Y_1, Y_2 of Y orthogonal each to other, a linear, injective operator $T: D \rightarrow Y_0$ and maps $A: D \rightarrow Y_1$, $B: D \rightarrow Y_2$ such that

$$\begin{aligned}D &\subset \text{im } T^*, \quad \text{cl im } T = Y_0, \\ f &= T + A, \quad g = (T^{-1})^* + B.\end{aligned}$$

Proof. (\implies) Assume that $f, g: D \rightarrow Y$ satisfy Eq. (1). In view of Theorem 9 we have the existence of the subspace Y_0 of $\text{cl Lin im } g$, linear, injective mappings $f_1, g_1: D \rightarrow Y_0$ and maps $f_2: D \rightarrow (\text{im } g)^\perp$, $g_2: D \rightarrow Y_0^\perp \cap \text{cl Lin im } g$ such that

$$\begin{aligned}\langle f_1(x)|g_1(y) \rangle &= \langle x|y \rangle, \quad x, y \in D, \\ f &= f_1 + f_2, \quad g = g_1 + g_2, \\ \text{cl im } f_1 &= \text{cl im } g_1 = Y_0.\end{aligned}$$

Let $Y_1 = (\operatorname{im} g)^\perp$, $Y_2 = Y_0^\perp \cap \operatorname{cl} \operatorname{Lin} \operatorname{im} g$, $A = f_2$, $B = g_2$, $T = f_1$. Hence T^{-1} exists and its domain $\operatorname{im} T = \operatorname{im} f_1$ is dense in Y_0 so there exists the adjoint operator $(T^{-1})^*$. We observe that

$$\langle T^{-1}(T(x))|y \rangle = \langle x|y \rangle = \langle f_1(x)|g_1(y) \rangle = \langle T(x)|g_1(y) \rangle, \quad x, y \in D,$$

so $D \subset D((T^{-1})^*)$ and $(T^{-1})^*(y) = g_1(y)$ for $y \in D$.

Hence, in view of Lemma 10 we get $D \subset \operatorname{im} T^*$ and also we have

$$\begin{aligned} f &= f_1 + f_2 = T + A, \\ g &= g_1 + g_2 = (T^{-1})^* + B, \\ \operatorname{cl} \operatorname{im} T &= \operatorname{cl} \operatorname{im} f_1 = Y_0. \end{aligned}$$

(\Leftarrow) Assume that there exist subspaces Y_0, Y_1, Y_2 of Y orthogonal to each other, a linear, injective operator $T: D \rightarrow Y_0$ and maps $A: D \rightarrow Y_1$, $B: D \rightarrow Y_2$ such that

$$\begin{aligned} D &\subset \operatorname{im} T^*, \quad \operatorname{cl} \operatorname{im} T = Y_0, \\ f &= T + A, \quad g = (T^{-1})^* + B. \end{aligned}$$

In view of Lemma 10 we get $D \subset D((T^{-1})^*)$ and we obtain

$$\begin{aligned} \langle f(x)|g(y) \rangle &= \langle T(x) + A(x)|(T^{-1})^*(y) + B(y) \rangle = \langle T(x)|(T^{-1})^*(y) \rangle \\ &= \langle T^{-1}(T(x))|y \rangle = \langle x|y \rangle, \quad x, y \in D. \end{aligned}$$

□

Now we show an example that the assumption $D \subset \operatorname{im} T^*$ is independent from other assumptions and cannot be omitted in the previous theorem.

Example 12. Let $\{e_n: n \in \mathbb{N}\}$ be a standard orthonormal base of ℓ^2 , $D = \{(x_n)_{n \in \mathbb{N}} \in \ell^2: \sum_{n=1}^{\infty} n^2 x_{2n}^2 < \infty\}$, $T: D \rightarrow \ell^2$ be given by the formula

$$T((x_n)_{n \in \mathbb{N}}) = \sum_{n=1}^{\infty} \frac{1}{n} x_{2n-1} e_{2n-1} + \sum_{n=1}^{\infty} n x_{2n} e_{2n}, \quad (x_n)_{n \in \mathbb{N}} \in D.$$

It is easy to see that D is a dense linear subspace of ℓ^2 , T is linear, injective and $\operatorname{im} T$ is dense in ℓ^2 (it contains the standard base).

Let $y \in D(T^*)$ then

$$\begin{aligned} &\sum_{n=1}^{\infty} \frac{1}{n} x_{2n-1} y_{2n-1} + \sum_{n=1}^{\infty} n x_{2n} y_{2n} \\ &= \langle T(x)|y \rangle = \langle x|T^*(y) \rangle = \sum_{n=1}^{\infty} x_n T^*(y)_n, \quad x \in D. \end{aligned}$$

Putting $x = e_k$ for $k \in \mathbb{N}$ we obtain

$$T^*(y) = \sum_{n=1}^{\infty} \frac{1}{n} y_{2n-1} e_{2n-1} + \sum_{n=1}^{\infty} n y_{2n} e_{2n}.$$

Let $y = \sum_{n=1}^{\infty} \frac{1}{n} e_{2n-1}$. Then $y \in D$. Suppose that $y = T^*(x)$ for some $x \in D(T^*)$. Hence

$$\sum_{n=1}^{\infty} \frac{1}{n} e_{2n-1} = y = T^*(x) = \sum_{n=1}^{\infty} \frac{1}{n} x_{2n-1} e_{2n-1} + \sum_{n=1}^{\infty} n x_{2n} e_{2n}.$$

Then we have $x_{2n-1} = 1$, $x_{2n} = 0$ for $n \in \mathbb{N}$ which gives a contradiction. So we obtain that $D \not\subset \text{im } T^*$.

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